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COFIBRATIONS IN THE BICATEGORY OF TOPOI

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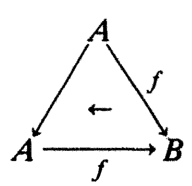
1. Introduction

A simple set of axioms for proarrow equipment on a bicategory $\mathscr{X}$ was introduced in [9]. A homomorphism of bicategories $(\ )_*:\mathscr{X}\to\mathscr{M}$ *equips* $\mathscr{X}$ *with proarrows* if it is locally fully faithful and for every arrow f in $\mathscr{X}$, f_* has a right adjoint f^* in $\mathscr{M}$. We may assume that the objects of $\mathscr{M}$ are those of $\mathscr{X}$ and $(\ )_*$ is the identity on objects. The motivating example is $(\ )_*:\mathrm{CAT}\to\mathrm{PROF}$ which associates to a functor $f:B\to A$ the profunctor $A(-,-f):B\to A$.

It is clear that there are variants of this example to handle $\mathscr{X}=\mathrm{cat}(S)$, $\mathscr{X}=$ S-indexed-CAT and $\mathscr{X}=V$-CAT for S with finite limits and monoidal V. In all three cases calculations with proarrows tend to become tedious and an axiomatic approach is labour-saving if nothing else. The third example is particularly interesting. Generally speaking, V-notions do not agree with the corresponding notions in V-CAT expressed in terms of CAT-valued representability. For example, to say "$f:A\to B$ is V-fully faithful" is usually stronger than to say "



is an absolute left lifting in V-Cat". (The latter is equivalent to saying that the underlying functor is fully faithful in CAT. See [7] or [9].) However, V-notions *can* be described bicategorically by referring to $(\ )_*:V\text{-CAT}\to V\text{-PROF}$.

A homomorphism $(\ )^*:\mathscr{X}^{\mathrm{coop}}\to\mathscr{M}$ is obtained when $f\mapsto f^*$ is extended to

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transformations (2-cells) by adjointness. It will often be convenient to regard this as $(\ )^*:\mathscr{K}^{op}\to\mathscr{M}^{co}$. Clearly $(\ )_*$ is essentially recoverable from $(\ )^*$, and $(\ )^*$ is locally fully faithful if and only if $(\ )_*$ is. For any bicategory $\mathscr{M}$, $f\dashv u$ in $\mathscr{M}$ if and only if $u\dashv f$ in $\mathscr{M}^{co}$. Hence:

Proposition 1. $(\ )_*:\mathscr{K}\to\mathscr{M}$ *equips* $\mathscr{K}$ *with proarrows if and only if* $(\ )^*:\mathscr{K}^{op}\to\mathscr{M}^{co}$ *equips* $\mathscr{K}^{op}$ *with proarrows.* □

A less dry example than those above was given in [9] following a suggestion of R. Paré. By $(\ )^*:\mathrm{TOP}^{op}\to\mathrm{LEX}$ we will mean "forget the direct image". Here TOP denotes the bicategory of topoi and geometric morphisms and LEX denotes the bicategory of topoi and left exact functors. Clearly this is proarrow equipment. $(\ )_*:\mathrm{TOP}\to\mathrm{LEX}^{co}$ necessarily becomes, *essentially*, "forget the inverse image", which brings us to a slight deviation from accepted terminology. By a geometric morphism $E\to F$ we mean a left exact functor $F\to E$ (between topoi) which *has* a right adjoint. With this convention the forgetful homomorphism $\mathrm{TOP}^{op}\to\mathrm{CAT}$ creates the limits in TOP^{op} which we will require and we do not have to speak of bilimits (a.k.a. pseudo-limits) as one does with the usual definition. (In spite of this convention, we accept the doctrine that binotions are the appropriate ones for 2-categories, even when notions are available.) We will write just f for a morphism in TOP^{op} that would normally be called f^*. A right adjoint of f will be denoted by f_*. Furthermore, we will denote by $\tilde{f}$ the unit for the adjunction $f\dashv f_*$, and $\underset{\sim}{f}$ will denote the counit.

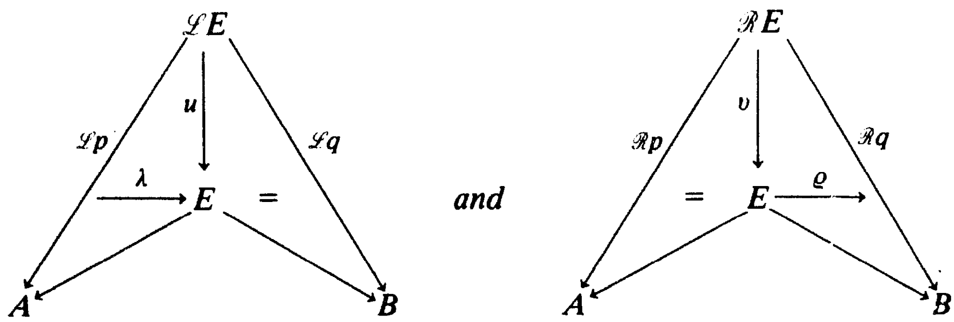
The idea that a left exact functor is a 'progeometric morphisms' has an intuitive appeal. For a left exact profunctor with a right adjoint is merely a left exact functor, modulo idempotents splitting in the codomain – a non-condition for a topos. It turns out, however, that the proarrow equipment $(\ )_*:\mathrm{TOP}\to\mathrm{LEX}^{co}$ enjoys more of the properties of $(\ )_*:\mathrm{CAT}\to\mathrm{PROF}$ than does $(\ )^*:\mathrm{TOP}^{op}\to\mathrm{LEX}$. The reason for this will become obvious in Section 7.

An axiomatic $(\ )_*:\mathscr{K}\to\mathscr{M}$ is not required to arise from universal constructions. For reasonable $\mathscr{K}$ Street has shown how to construct $\mathscr{M}=\mathrm{PROF}(\mathscr{K})$ using fibrations in $\mathscr{K}$. The reader is referred to [6]. Our main result here is that $\mathrm{LEX}^{co}(B,A)$ is equivalent to the category of codiscrete TOP cofibrations from B to A, i.e. $\mathrm{LEX}^{co}(B,A)\simeq\mathrm{PROF(TOP)}(B,A)$ according to Street's constuction. We arrive at this by characterizing arbitrary TOP cofibrations using certain diagrams in LEX^{co}.

2. Cofibrations in TOP

By definition, a cofibration in TOP is a fibration in TOP^{op}.

Lemma 2. *For a span* $p:A\leftarrow E\to B:q$ *in* TOP^{op} *there exist universal diagrams*

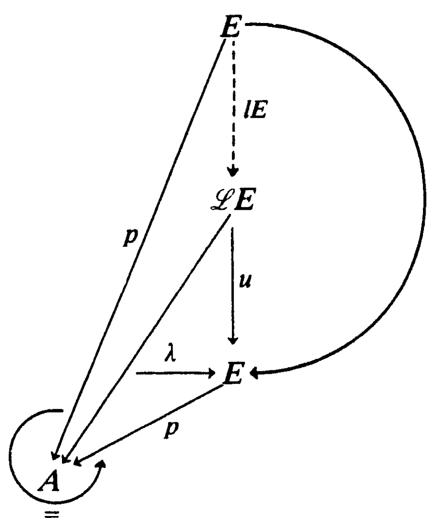


in TOP^{op}.

Proof. It is well known that TOP has finite lax colimits. See, for example, 4.25 of [3]. Explicitly, take A/p for $\mathscr{L}E$ with the projections for $\mathscr{L}p$ and u. For $\mathscr{R}E$ take $q/B \xrightarrow{\sim} E/q_*$. The isomorphism identifies projections.) □

Now, following Street [6], $\mathscr{R}$ underlies a KZ doctrine and $\mathscr{L}$ underlies a coKZ doctrine on (the bicategory) (SPN TOP^{op})(B, A). Briefly, $\mathscr{L}$ is a monad for which any algebra $((p, E, q), @ : \mathscr{L}E \to E)$ enjoys $lE \dashv @$ in (SPN TOP^{op})(B, A), where l is the unit for $\mathscr{L}$. Similarly, for any structure $@ : \mathscr{R}E \to E$, $@ \dashv rE$, where r is the unit for $\mathscr{R}$. So carrying an $\mathscr{L}$- or $\mathscr{R}$-algebra 'structure' is a property of a span. In Street's terminology: A span $p : A \leftarrow E \to B : q$ is a TOP^{op} *left* (respectively *right*) *fibration from B to A* if and only if it is an $\mathscr{L}$- (respectively $\mathscr{R}$-) algebra.

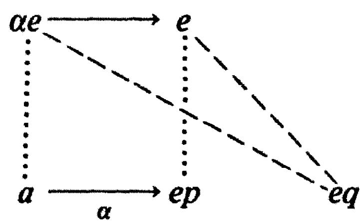
It is crucial to observe the bicategories in which the above adjunctions take place. lE is defined by



$(e \mapsto (ep \to ep, e))$. Lemma 2 and this definition ensure that lE is an arrow of TOP^{op} and moreover that it is an arrow of spans. So lE has a CAT right adjoint, $(lE)_*$, and for future convenience we note that it is given by

$$\begin{array}{ccc} (\alpha : a \to ep, e) \mapsto (\alpha, e)(lE)_* & \longrightarrow & e \\ \Big\downarrow & \times & \Big\downarrow{\scriptstyle e\tilde{p}} \\ ap_* & \xrightarrow[ap_*]{} & epp_* \end{array}$$

where the square is a pullback and $\tilde{p}$ denotes the unit for the CAT adjunction $p \dashv p_*$. $(lE)_*$ is not necessarily an arrow between the underlying CAT spans. This would require precisely that $(\alpha,e)(lE)_*p = a$ and $(\alpha,e)(lE)_*q = eq$, which would mean precisely that the underlying CAT span of (p, E, q) is a CAT left fibration from B to A. In such a situation we write $\alpha e \to e$ for $(\alpha,e)(lE)_* \to e$ and use a suggestive notation that seems to be due to Benabou:



We also, in this situation, refer to $(lE)_*$ as a left action and say that $\alpha e \to e$ is *cartesian over* α. It is easily seen that $f \dashv g$ in $\mathrm{TOP}^{\mathrm{op}}$ means that $f_* \simeq g$. So to say that $lE \dashv @$ in $(\mathrm{SPN}\ \mathrm{TOP}^{\mathrm{op}})(B, A)$ is to say that $(lE)_* \simeq @$ is as above and has a CAT right adjoint, $@_*$ (which is not necessarily an arrow of spans). Summarizing:

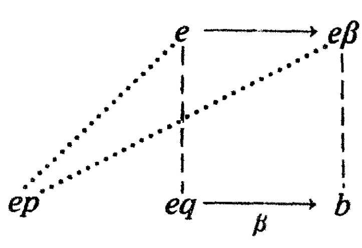
Proposition 3. *A span* $p : A \leftarrow E \rightarrow B : q$ *in* $\mathrm{TOP}^{\mathrm{op}}$ *is a* $\mathrm{TOP}^{\mathrm{op}}$ *left fibration from B to A if and only if it is a* CAT *left fibration from B to A and the left action has a* CAT *right adjoint.* □

We will interpret such $@_*$ later. Note that we always have a $\mathrm{TOP}^{\mathrm{op}}$ adjunction $u \dashv lE$ and a CAT adjunction $u_! \dashv u$. $eu_! = (! : 0 \to ep, e)$; so $u_!$ does not preserve 1 and is thus not an arrow in $\mathrm{TOP}^{\mathrm{op}}$. (It clearly does preserve pullbacks.)

Mutatis mutandis we have:

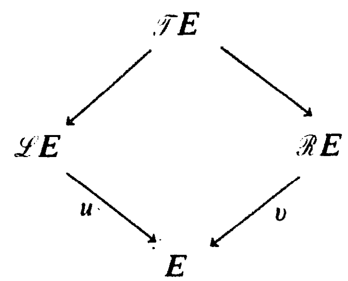
Proposition 4. *A span* $p : A \rightarrow E \leftarrow B : q$ *in* $\mathrm{TOP}^{\mathrm{op}}$ *is a* $\mathrm{TOP}^{\mathrm{op}}$ *right fibration from B to A if and only if it is a* CAT *right fibration from B to A and the right action is left exact.* □

The term 'right action' and our notation is best summed up by:

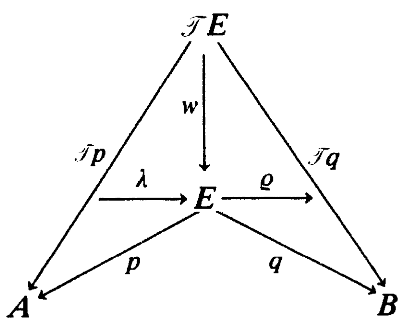


The reader is assumed to be familiar with the universal property that a (SPN CAT)(B, A) adjunction, $@ \dashv rE$, gives to $e\beta := (e, \beta)@$. A standard reference is [2]. Since the actions are necessarily unitary, left exactness of @ just means that it preserves pullbacks.

As they must, $\mathscr{R}\mathscr{L} \simeq \mathscr{L}\mathscr{R}$, and we write $\mathscr{T}$ for the doctrine built from this distributive law. One might note in passing that



is a pullback in TOP^{op} created by TOP$^{op} \to$ CAT, a pushout of inclusions in TOP. Objects of $\mathscr{T}E$ are just triples $(\alpha : a \to ep, e, \beta : eq \to b)$ where α, β are morphisms in A, B and e is in E. Given $p : A \leftarrow E \to B : q$ in TOP^{op},



where $\mathscr{T}p$, w and $\mathscr{T}q$ denote projections; is a universal such diagram in TOP^{op}. A TOP^{op} *fibration from B to A* is a $\mathscr{T}$-algebra, that is, a span which is simultaneously a left fibration and a right fibration such that the actions associate [6]. The latter means that $(\alpha e)\beta \rightharpoonup \alpha(e\beta)$ generically, and we will see later that associativity follows from the other data in the TOP^{op} case.

Corollary 5. *A span $p : A \leftarrow E \to B : q$ is a* TOP^{op} *fibration from B to A if and only if it is a* CAT *fibration from B to A and the left action has a* CAT *right adjoint and the right action is left exact.* □

Discreteness in a bicategory can be defined in terms of $(\ )^2$, when such limits exist. The forgetful homomorphism TOP$^{op} \to$ CAT creates $(\ )^2$ from which it follows that (SPN TOP^{op})$(B, A) \to$ (SPN CAT)(B, A) does too. Whence:

Corollary 6. *Corollary 5 remains valid when 'fibration' is replaced by 'discrete fibration'.* □

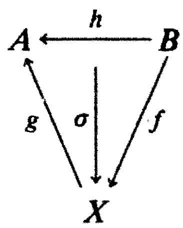
So, for topoi A and B, among all functors $A^{op} \times B \to SET$ are those for which the corresponding discrete fibration has a two-sided action which is left exact with a right adjoint. It is natural to conjecture that these give rise to profunctors $B \to A$ which are left exact and have right adjoints. With obvious notation:

Theorem 7. (DFIB TOP^{op})$(B, A) \sim$ LEX(B, A).

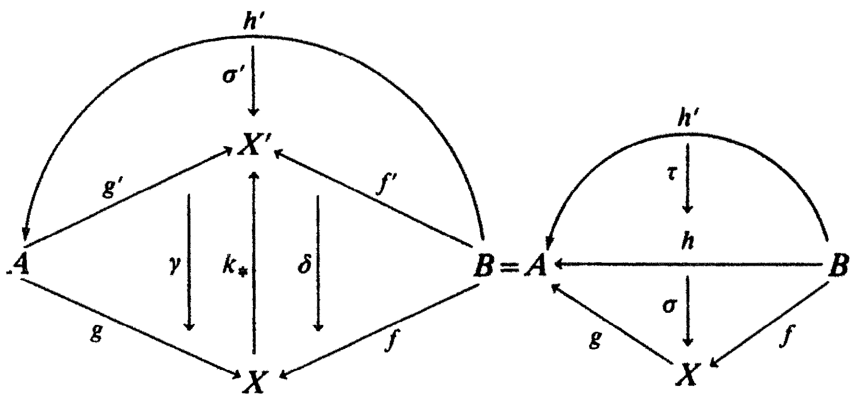
Proof. We prefer to prove this as a corollary of our characterization of not-necessarily-discrete fibrations (Theorem 24). □

3. Gamuts relative to ( )$_*$: TOP→LEX^{co}

In [6], Street introdued *V-gamuts* in his characterization of cofibrations in *V*-CAT. Gamuts and related notions can be defined in the context of any proarrow equipment $(\ )_* : \mathscr{K} \to \mathscr{M}$. We limit ourselves to $(\ )_* :$ TOP→LEX^{co} here. For topoi A and B, a ( )$_*$-*gamut* (or simply *gamut*) *from B to A* is a diagram:



in LEX. An arrow from (h, X, f, g, σ) to $(h', X', f', g', \sigma')$ consists of a transformation $\tau : h' \to h$ (in LEX), a TOP arrow $k_* : X \to X'$ and transformations $\gamma : k_* g' \to g$ and $\delta : f' \to f k_*$ such that



A transformation from $(\tau, k_*, \gamma, \delta)$ to $(\tau', k'_*, \gamma', \delta')$ is a transformation $\kappa_* : k'_* \to k_*$ satisfying $\kappa_* g' \cdot \gamma = \gamma'$ and $\delta' \cdot f\kappa_* = \delta$. We write $*$GAM(B, A) for the resulting bicategory.

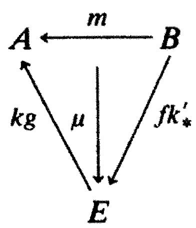
It is easy to see that if X is **1**, the zero object of LEX^{co}, then $(h, \mathbf{1}, !, !, !)$ is *codiscrete* in $*$GAM(B, A). (Recall that an object C of a bicategory is codiscrete if any transformation

$$C \underset{\longrightarrow}{\overset{\longrightarrow}{\downarrow}} T$$

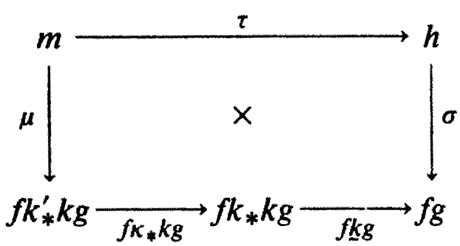
is necessarily an identity transformation.) We can establish the converse. As usual we cloud the distinction between the notion, in this case codiscreteness, and the more appropriate binotion.

Lemma 8. *If (h, X, f, g, σ) is codiscrete in* $*$GAM(B, A), *then X is codiscrete in* TOP.

Proof. Let $\kappa_* : k'_* \to k_*$; $k'_*, k_* : X \rightrightarrows E$ be determined by a TOP transformation. Consider the gamut,



where

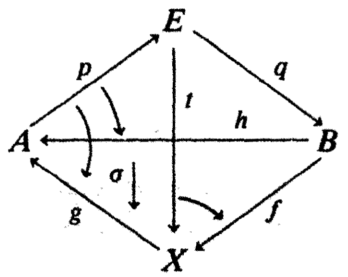


is a pullback in LEX(B, A), $\underset{\sim}{k}$ being the counit for the CAT adjunction, $k \dashv k_*$. It is now easy to verify that $(\tau, k_*, \underset{\sim}{k}g, f\kappa_*)$ and $(\tau, k'_*, \kappa_*kg \cdot \underset{\sim}{k}g, fk'_*)$ are arrows from (h, X, f, g, σ) to (m, E, fk'_*, kg, μ) and that κ_* is a transformation between them in $*$GAM(B, A). Hence, κ_* is an identity. □

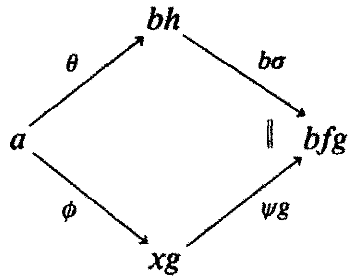
Since TOP$^{op} \to$ CAT creates $(\ )^2$, **1** is essentially the only codiscrete in TOP. Writing (COD$*$GAM)(B, A) for the bifull subbicategory of ($*$GAM)(B, A) determined by the codiscretes we have:

Proposition 9. (COD$*$GAM)$(B, A) \sim$ LEX$^{co}(B, A)$. □

A gamut from B to A, (h, X, f, g, σ), is a unitary morphism of bicategories, $\mathbf{3} \to$ LEX. We can apply Wraith's 'glueing construction' [10] to a gamut and get a universal diagram, where the constructed arrows p, q and t are in TOP^{op}:



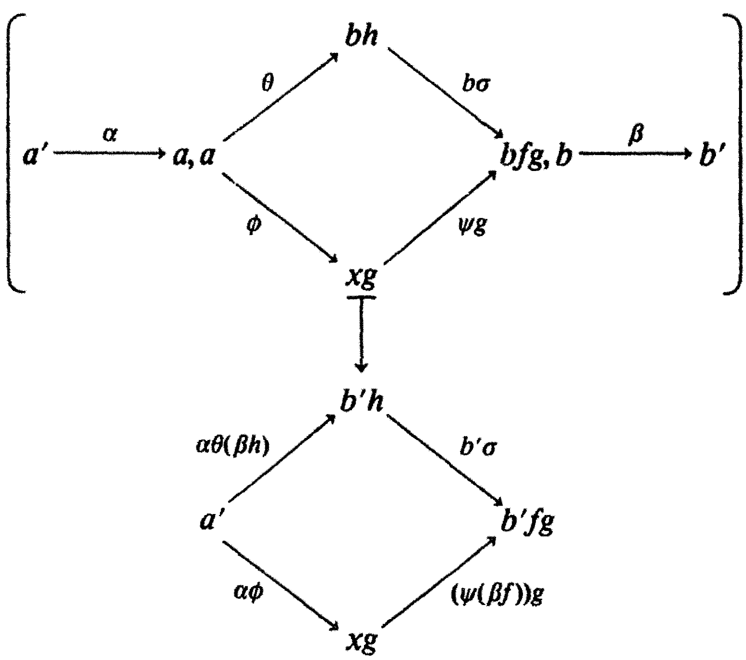
The objects of E are 6-tuples $(a, x, b, \theta, \phi, \psi)$, where $a \in A$, $x \in X$, $b \in B$ and θ, ϕ, and ψ are morphisms in A satisfying



The morphisms of E are as expected.

Proposition 10. *The* TOP^{op} *span* $p: A \leftarrow E \rightarrow B: q$ *constructed above is a* TOP^{op} *fibration from* B *to* A.

Proof. The action $\mathcal{T}E \rightarrow E$ is best described symbolically by



It is entirely straightforward to show that this makes (p, E, q) a CAT fibration from

B to A. The right action is left exact since h, f and g are. The left action has a right adjoint which sends our generic object of E to

$$\begin{array}{ccccc} & & bh & & \\ & \nearrow & & \searrow^{b\sigma} & \\ a\to\bar{a},\ \bar{a} & & \times & & bfg \\ & \searrow & & \nearrow_{\psi g} & \\ & & xg & & \end{array}$$

where the square is now a pullback (in A) and $a\to\bar{a}$ is the fill-in determined by the original square. We omit the rather tedious verification of this. □

Proposition 11. *The glueing construction extends to a homomorphism of bicategories* $\Sigma : *\mathrm{GAM}(B,A)\to(\mathrm{FIB\ TOP}^{\mathrm{op}})(B,A)^{\mathrm{op}}=(\mathrm{COFIB\ TOP})(B,A)$. □

Remark 12. As usual, an arrow between fibrations is a $\mathscr{T}$-algebra homomorphism. The apparently bizarre definition of arrows between gamuts is precisely what is needed to establish Proposition 11. We note, however, that there are independent considerations that lead one to such a definition (cf. Thiébaud [8]). □

We will show in the next section that all $\mathrm{TOP}^{\mathrm{op}}$ fibrations from B to A essentially arise from the glueing construction.

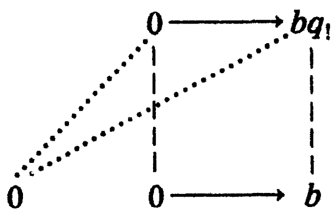
Given a $\mathrm{TOP}^{\mathrm{op}}$ fibration $p:A\leftarrow E\to B:q$ we define $t:E\to E$ by $et=(e!)@_*u$. By $e!$ we mean $e\beta$ as above where $\beta=!:eq\to 1$ in B. Consider

$$\begin{array}{c} e\to et \\ \hline e\to (e!)@_*u \\ \hline eu_!@\to e! \\ \hline !e\to e! \end{array}$$

Here $!e$ is αe, where $\alpha=!:0\to ep$. Note that we are using $eu_!@\simeq !e$, which follows immediately from the description of $u_!$ given earlier. We define $e\tilde{t}:e\to et$ to be the transpose of $!e\to e\to e!$. Our immediate goal is to show that t together with $\tilde{t}$ is an idempotent left exact triple on E.

Lemma 13. *q has a fully faithful* CAT *left adjoint, $q_!$, and $q_!p=0$.*

Proof. Since $q:E\to B$ is a CAT right fibration (from B to **1**), $e/E\to eq/B$ has a fully faithful left adjoint, for all e in E. In particular $q:E\simeq 0/E\to 0q/B\simeq B$ has a left adjoint, $q_!$. Since $p:A\leftarrow E\to B:q$ is a CAT fibration from B to A, $q_!p=0$ follows from:



$\square$

Corollary 14. $!(bq_!) \simeq bq_!$, *for all b in B.* $\square$

Lemma 15. $@_*uq \simeq q$.

Proof.

$$\begin{array}{c} b \to e@_*uq \\ \hline (bq_!)u_!@ \to e \\ \hline !(bq_!) \to e \\ \hline bq_! \to e \\ \hline b \to eq \end{array} \qquad \square$$

Proposition 16. *t together with $\tilde{t}$ is an idempotent left exact triple on E.*

Proof. t is left exact since the right action is.

$$\begin{array}{rl} & d \to ett \\ \hline & d \to (((e!)@u)!)@_*u \\ \hline & !d \to ((e!)@_*u)! \\ \hline \dagger & !d \to (e!)@_*u \\ \hline & !!d \to e! \\ \hline & !d \to e! \\ \hline & d \to et \end{array}$$

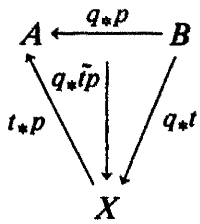
For $\dagger$ above we note that

$$((e!)@_*u)! \simeq (e!)@_*u,$$

since

$$(e!)@_*uq \simeq (e!)q \simeq 1. \quad \square$$

Let X denote the subtopos of E corresponding to t. We abuse notation and write $t: E \to X$ in $\mathrm{TOP}^{\mathrm{op}}$. Now



defines a gamut from B to A.

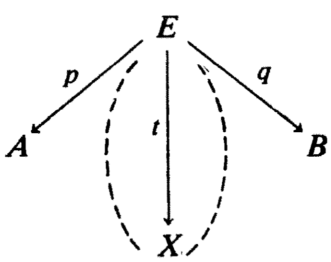
Proposition 17. *The construction above extends to a homomorphism of bicategories* $\Delta : (\mathrm{COFIB\ TOP})(B, A) \to {*}\mathrm{GAM}(B, A)$. □

4. The main result

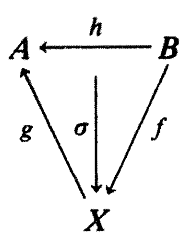
We will show that Σ and Δ introduced in Section 3 constitute a biequivalence, ${*}\mathrm{GAM}(B, A) \sim (\mathrm{COFIB\ TOP})(B, A)$.

Proposition 18. ${*}\mathrm{GAM}(B, A) \to (\mathrm{COFIB\ TOP})(B, A) \to {*}\mathrm{GAM}(B, A)$ *is the identity.*

Proof. If



is Σ of



then:

(i) We may take

$$bq_* = \left[\begin{array}{ccccc} & & bh & & \\ & \overset{=}{\nearrow} & & \overset{b\sigma}{\searrow} & \\ bh & & & & bfg \\ & \underset{b\sigma}{\searrow} & & \underset{=}{\nearrow} & \\ & & (bf)g & & \end{array}\right]$$

Hence $q_*p = h$ and $q_*t = f$.

(ii) We may take

$$xt_* = \left[\begin{array}{ccccc} & & 1h & & \\ & \overset{!}{\nearrow} & & \overset{1\sigma}{\searrow} & \\ xg & & & & 1fg \\ & \underset{=}{\searrow} & & \underset{!}{\nearrow} & \\ & & xg & & \end{array}\right]$$

Hence $t_*p = g$.

(iii) From (ii) we see that

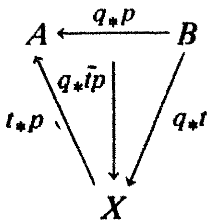
$$d = \left[\begin{array}{ccccc} & & bh & & \\ & \overset{\theta}{\nearrow} & & \overset{b\sigma}{\searrow} & \\ a & & & & bfg \\ & \underset{\phi}{\searrow} & & \underset{\psi g}{\nearrow} & \\ & & xg & & \end{array}\right] \xrightarrow[d\tilde{t} = (\phi, !, x)]{} \left[\begin{array}{ccccc} & & 1h & & \\ & \overset{!}{\nearrow} & & \overset{1\sigma}{\searrow} & \\ xg & & & & 1fg \\ & \underset{=}{\searrow} & & \underset{!}{\nearrow} & \\ & & xg & & \end{array}\right] = dtt_*$$

which together with the diagram in (i) yields $q_*\tilde{t}p = \sigma$. □

To establish that $(\text{COFIB TOP})(B, A) \to {*}\text{GAM}(B, A) \to (\text{COFIB TOP})(B, A)$ is *equivalent* to the identity we find it convenient to introduce some more notation. We write $(\bar{p}, \bar{E}, \bar{q})$ for $\Sigma(\Delta(p, E, q))$ and $k : E \to \bar{E}$ for the functor defined by

$$e \mapsto \begin{array}{ccccc} & & eqq_*p & & \\ & \overset{e\bar{q}p}{\nearrow} & & \overset{eqq_*\tilde{t}p}{\searrow} & \\ ep & & & & eqq_*tt_*p \\ & \underset{e\tilde{t}p}{\searrow} & & \underset{e\bar{q}tt_*p}{\nearrow} & \\ & & ett_*p & & \end{array}$$

(As before we are writing



for $\Delta(p,E,q)$.) k is clearly an arrow of spans. It is also an arrow over X, if we write $\bar{t}:\bar{E}\to X$ for the other projection.

Now, it is well known from Wraith's glueing construction that $\bar{E}$ is isomorphic to the category of coalgebras for a left exact cotriple on $A\times X\times B$, by an isomorphism that identifies $(\bar{p},\bar{t},\bar{q})$ with the forgetful functor. Our present task amounts to showing that if (p,E,q) is a TOP$^{\text{op}}$ fibration, then $(p,t,q):E\to A\times X\times B$ is cotripleable. $k:E\to\bar{E}$ is the usual comparison functor. The next lemma will allow us to simplify notation somewhat.

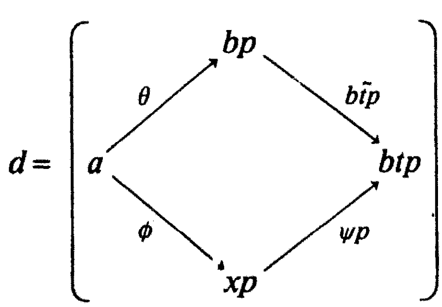
Lemma 19. *If (p,E,q) is a* TOP$^{\text{op}}$ *fibration, then p_*, t_* and q_* are inclusions in* TOP.

Proof. t_* is so by construction. From Lemma 13 we have $q_!$ fully faithful and since $q_!\dashv q\dashv q_*$ in CAT we have q_* fully faithful also. Since $p:E\to A$ is a CAT left fibration (from **1** to A), $E/e\to A/ep$ has a fully faithful right adjoint, for all e in E. By taking $e=1$ we see that p_* is fully faithful. □

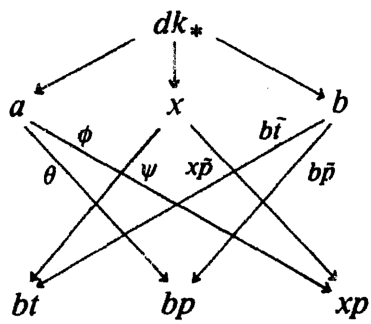
This allows us to suppress p_* and write $\tilde{p}:E\to p$ for $\tilde{p}:E\to pp_*$, when convenient. We do similarly for t_* and q_*.

Lemma 20. *$k:E\to\bar{E}$ has a right adjoint, k_*.*

Proof. Actually, this together with left exactness of k follows because our constructions ensure that k is a TOP$^{\text{op}}$ arrow. For



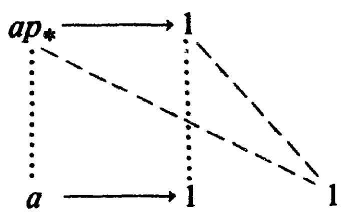
in $\bar{E}$, in which q_* and t_* are suppressed, dk_* is given by the following inverse limit diagram in E:



Verification of this is straightforward. □

Lemma 21. *For (p, E, q) a* TOP^{op} *fibration,* $p_*q \simeq 1$, $p_*t \simeq 1$ *and* $t_*q \simeq 1$.

Proof. (i) $p_*q \simeq 1$ follows from

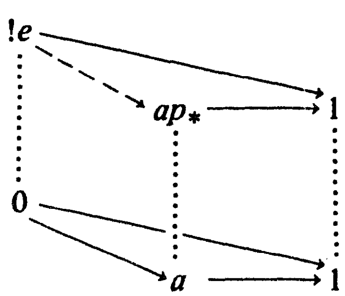


after the proof of Lemma 19.

(ii) For $p_*t \simeq 1$ observe that in E

$$\frac{\dfrac{\dfrac{e \to ap_*t}{e \to ((ap_*)!)@_*u}}{!e \to (ap_*)!}}{!e \to ap_*}\ \dagger$$

where † follows from (i). Since



and the front square is cartesian there is exactly one such morphism in E.

(iii) For $t_*q \simeq 1$ it suffices to show $(e!)@_*uq \simeq 1$, for all e in E. This follows immediately from Lemma 15. □

Lemma 22. k_* *is fully faithful.*

Proof. We show $\underset{\sim}{k}: k_*k \xrightarrow{\sim} \bar{E}$. This requires that we have isomorphisms, $dk_*p \xrightarrow{\sim} a$, $dk_*t \xrightarrow{\sim} x$ and $dk_*q \xrightarrow{\sim} b$ satisfying

$$\begin{array}{ccc} dk_*p & \xrightarrow{\ \cong\ } & a \\ {\scriptstyle dk_*\tilde{q}p}\downarrow & & \downarrow{\scriptstyle \theta} \\ dk_*qp & \xrightarrow{\ \cong\ } & bp \end{array} \qquad \begin{array}{ccc} dk_*p & \xrightarrow{\ \cong\ } & a \\ {\scriptstyle dk_*\tilde{t}p}\downarrow & & \downarrow{\scriptstyle \phi} \\ dk_*tp & \xrightarrow{\ \cong\ } & xp \end{array} \quad \text{and} \quad \begin{array}{ccc} dk_*t & \xrightarrow{\ \cong\ } & x \\ {\scriptstyle dk_*\tilde{q}t}\downarrow & & \downarrow{\scriptstyle \psi} \\ dk_*qt & \xrightarrow{\ \cong\ } & bt \end{array}$$

for all d, as above, in $\bar{E}$. Such isomorphisms are found when p, t and q are applied to the finite limit diagram which defines k_* and the results of Lemma 21 are taken into account. □

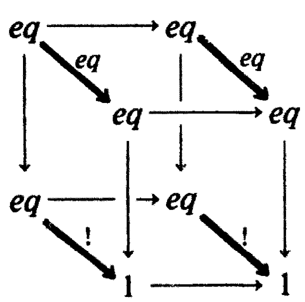
Lemma 23. *For all e in E, the following diagram is a pullback in E*:

$$\begin{array}{ccc} e@_*u & \longrightarrow & eq \\ \downarrow & & \downarrow{\scriptstyle eq\tilde{t}} \\ et & \xrightarrow[e\tilde{q}t]{} & eqt \end{array}$$

Proof. Since

$$\begin{array}{ccc} e & \xrightarrow{e\tilde{q}} & eq \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ e & \xrightarrow[e\tilde{q}]{} & eq \end{array}$$

is a pullback in E and



is a pullback in B^2 and the right action is left exact,

$$\begin{array}{ccc} e = e(eq) & \longrightarrow & eq(eq) = eq \\ \downarrow & & \downarrow \\ e! & \xrightarrow[(e\tilde{q})!]{} & (eq)! \end{array}$$

is a pullback in E. Applying $@_*u$ and noting the definition of t we have a pullback,

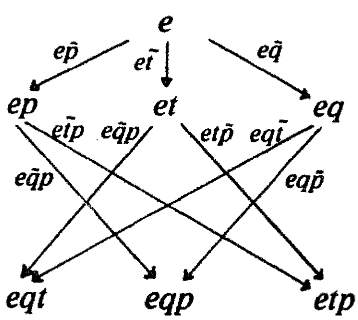
$$\begin{array}{ccc} e@_*u & \longrightarrow & eq@_*u \\ \downarrow & & \downarrow \\ et & \xrightarrow[e\bar{q}t]{} & eqt \end{array}$$

But $q@_*u \dashrightarrow q$ as the following calculation shows:

$$\begin{array}{c} d \to eq@_*u \text{ in } E \\ \hline !d \to eq \quad \text{in } E \\ \hline (!d)q \to eq \quad \text{in } B \\ \hline dq \to eq \quad \text{in } B \\ \hline d \to eq \quad \text{in } E \end{array} \qquad \square$$

Theorem 24. (COFIB TOP)$(B, A) \sim *$GAM(B, A).

Proof. After Proposition 18 and Lemma 22, it only remains to show $\tilde{\kappa}: E \dashrightarrow kk_*$, i.e. the diagram



is a limit diagram in E, for all e in E.

Since lE is fully faithful and $lE \dashv @ \dashv @_*$, $@_*$ is fully faithful. From this we see that $e@_*$ is of the form $(e\kappa : ep \to e@_*up, e@_*u)$ and that $e \simeq (e\kappa)e@_*u$, for all e in E. In our simplified notation the last isomorphism can be expressed by the following pullback diagram in E:

$$\begin{array}{ccc} e & \longrightarrow & e@_*u \\ {\scriptstyle e\tilde{p}}\downarrow & & \downarrow{\scriptstyle e@_*u\tilde{p}} \\ ep & \xrightarrow[e\kappa]{} & e@_*up \end{array}$$

This pullback together with that of Lemma 23 enable one to show that e is the required limit. □

Corollary 25 (Theorem 7). (CODCOFIB TOP)$(B, A) \sim \mathrm{LEX}^{\mathrm{co}}(B, A)$.

Proof. Proposition 9. (Note that $(\mathrm{COFIB}\,\mathscr{K})(B, A) = (\mathrm{FIB}\,\mathscr{K}^{\mathrm{op}})(B, A)^{\mathrm{op}}$ and $\mathscr{K}^{\mathrm{co}}(B, A) = \mathscr{K}(B, A)^{\mathrm{op}}$.) □

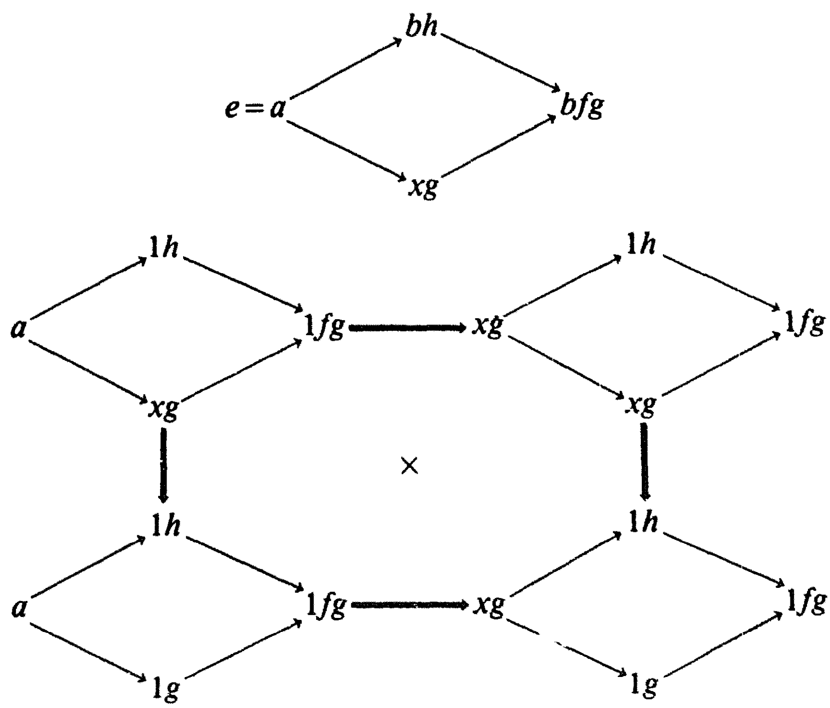
5. Further aspects of TOP cofibrations

Theorem 24 enables one to quickly discover many properties of $\mathrm{TOP}^{\mathrm{op}}$ fibrations. Since all essentially arise from gamuts, 'diagram lemmas' are particularly easy to check. A single example (cf. Lemma 23) should illustrate the point.

Lemma 26. *For all e in E, (p, E, q) a* $\mathrm{TOP}^{\mathrm{op}}$ *fibration, the following diagram is a pullback in E:*

$$\begin{array}{ccc} e! & \longrightarrow & et \\ \Big\downarrow & & \Big\downarrow{\scriptstyle et\tilde{p}} \\ ep & \xrightarrow[e\bar{t}p]{} & etp \end{array}$$

Proof. Let



□

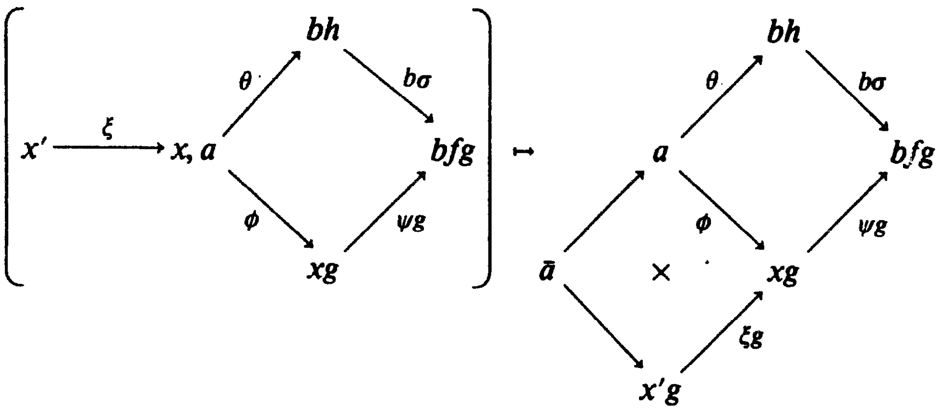
A $\mathrm{TOP}^{\mathrm{op}}$ fibration, (p, E, q), gives rise to two more (CAT) actions on E.

Lemma 27. *For any $s: E \to T$ in* CAT; *if $s \dashv s_*$ with s_* fully faithful and if E has and s preserves pullbacks, then s is a* CAT *left fibration (from* $\mathbf{1}$ *to T).* □

So, for (p, E, q) a $\mathrm{TOP}^{\mathrm{op}}$ fibration, $t: E \to X$ and $q: E \to B$ are also CAT left fibrations. We can say somewhat more.

Proposition 28. *$t: X \leftarrow E \to B: q$ is a* LEX *fibration from B to X.*

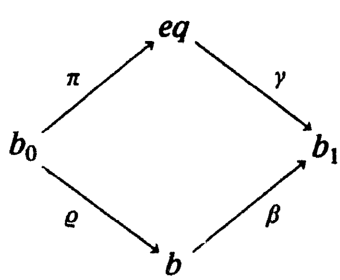
Proof. LEX→CAT creates the limits needed to define $\mathscr{L}$, $\mathscr{R}$ and $\mathscr{T}$ as before. If (p, E, q) arises from the gamut (h, X, f, g, σ), it is easily verified that



describes the left X action. Clearly this is compatible with the original right B action and the two associate. □

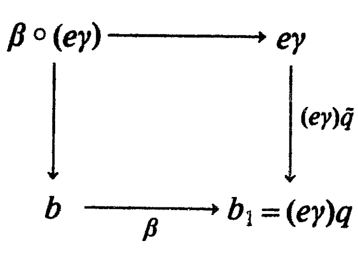
For e in E and $\beta: b \to eq$ in B, we write $\beta \circ e$ for the *left* action of B on E.

Proposition 29. *$q: E \to B$ satisfies the Beck–Chevalley condition. That is, if*



is a pullback in B, then $(\pi \circ e)\varrho \xrightarrow{\sim} \beta \circ (e\gamma)$.

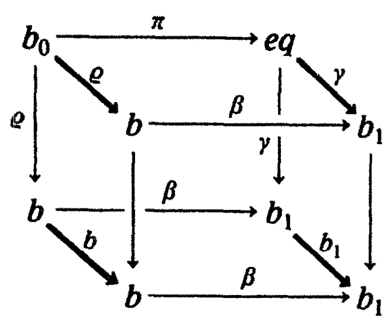
Proof. $- \circ -$ can be calculated using pullbacks in E. We have



a pullback in E (suppressing q_*). But also,

$$\begin{array}{ccc} \pi\circ e & \longrightarrow & e \\ \big\downarrow & & \big\downarrow{\scriptstyle e\tilde{q}} \\ b_0 & \xrightarrow{\ \pi\ } & eq \\ {\scriptstyle \varrho}\big\downarrow & & \big\downarrow{\scriptstyle \gamma} \\ b & \xrightarrow[\ \beta\ ]{} & b_1 \end{array}$$

is a pullback in E and

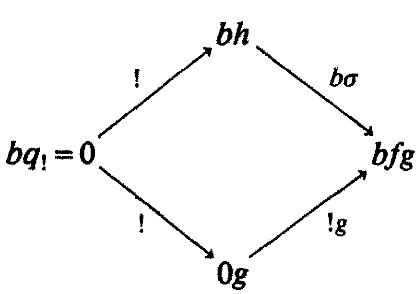


is a pullback in B^2, whence

$$\begin{array}{ccc} (\pi\circ e)\varrho & \longrightarrow & e\gamma \\ \big\downarrow & & \big\downarrow \\ b & \longrightarrow & b_1 \end{array}$$

is a pullback and $(\pi\circ e)\varrho \stackrel{\sim}{\to} \beta\circ(e\gamma)$. □

If (p,E,q) arises from a gamut, (h,X,f,g,σ), it is easy to see that $q_!$ can be described by



Proposition 30. (i) *$q_!$ preserves and reflects pullbacks.*

(ii) *For all e in E and all b in B, $(eq\times b)q_! \stackrel{\sim}{\to} e\times bq_!$.* □

Corollary 31. *$q: E \to B$ is a logical morphism.*

Proof. See Theorem E of [1]. □

Corollary 32. *$q_*: B \to E$ is an open inclusion in* TOP *if and only if it is a* TOP *right cofibration (from B to* **1***).*

Proof. 'If' follows from Corollary 31 and Lemma 1.4 of [4]. Conversely, if q_* is an open inclusion, we may take $p_*: A \to E$ to be the complementary closed inclusion and the glueing construction for a left exact functor tells us that (p_*, E, q_*) is a TOP codiscrete cofibration from B to A. In particular q_* is a TOP right cofibration. □

Of course the 'if' part above also follows immediately from the observation that $q_*: B \to E$ is equivalent to $E/\upsilon \to E$, where $\upsilon \rightarrowtail 1$ in E (for E arising from a gamut) has

$$\upsilon = \left[\begin{array}{ccc} & 1h & \\ {}^{!}\nearrow & & \searrow^{1\sigma} \\ 0 & & 1fg \\ {}_{!}\searrow & & \nearrow_{!g} \\ & 0g & \end{array}\right].$$

Similarly, for $p_*: A \to E$ arising from a gamut, we see that p_* is the closed inclusion corresponding to $u \rightarrowtail 1$ in E, where

$$u = \left[\begin{array}{ccc} & 1h & \\ {}^{!}\nearrow & & \searrow^{1\sigma} \\ 0 & & 1fg \\ {}_{!}\searrow & & \nearrow_{1g} \\ & 1g & \end{array}\right].$$

Proposition 33. *$p_*: A \to E$ is a closed inclusion in* TOP *if and only if it is a TOP left cofibration (from* 1 *to A).* □

For a TOP cofibration (p_*, E, q_*), with u and υ as above, we see that $u \leftarrowtail \upsilon$. This expresses the fact that p_* and q_* are 'disjoint'. Cofibrations can be constructed from such data:

Proposition 34. *If $1 \leftarrowtail u \leftarrowtail \upsilon$ in a topos E and $p_*: A \to E$ is the closed inclusion corresponding to u and $q_*: B \to E$ is the open inclusion corresponding to υ, then (p_*, E, q_*) is a* TOP *cofibration from B to A.*

Proof. After Corollary 32 and Proposition 33 we have only to check the compatibility of the actions and their association. We have A equivalent to the full subcategory of E determined by those e for which $e \times u \xrightarrow{\sim} u$, $B(\sim E/\upsilon)$ equivalent to the full subcategory of E determined by those e for which $e \xrightarrow{\sim} e^{\upsilon}$.

$$\begin{array}{ccc} e \times u & \longrightarrow & u \\ \downarrow & & \downarrow \\ e & \xrightarrow[e\bar{p}]{} & ep \end{array}$$

is a pushout and $e\tilde{q} : e \to eq$ is the diagonal, $e \to e^{\upsilon}$. Now pq is 1, since u is initial in A and $\upsilon \rightarrowtail u$. For e in E and $\alpha : a \to ep$ in $A, \alpha e$ is given by the following pullback in E:

$$\begin{array}{ccc} \alpha e & \longrightarrow & e \\ \downarrow & \times & \downarrow{\scriptstyle e\bar{p}} \\ a & \xrightarrow[\alpha]{} & ep \end{array}$$

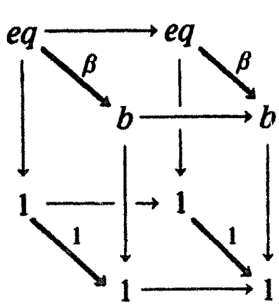
After applying $q = (\ )^{\upsilon}$ to this we still have a pullback, but with bottom row $1 \to 1$. Hence $(\alpha e)q \simeq eq$ and (p_*, E, q_*) is a left cofibration from B to A.

For $\beta : eq \to b$ in B, let $\bar{\beta} : e \to b$ denote its transpose in E. Since q has a left adjoint, viz. $- \times \upsilon$, and $e^{\upsilon} \times \upsilon \xrightarrow{\sim} e \times \upsilon$, we can calculate $e\beta$ as the following pushout in E:

$$\begin{array}{ccc} e \times \upsilon & \xrightarrow{\bar{\beta} \times \upsilon} & b \times \upsilon \\ \downarrow & & \downarrow \\ e & \longrightarrow & e\beta \end{array}$$

Since p preserves pushouts, $ep \xrightarrow{\sim} (e\beta)p$ if p inverts the top row. But for any e we have $(e \times \upsilon)p \xrightarrow{\sim} u$, since $\upsilon \rightarrowtail u$ implies $u \times \upsilon \xrightarrow{\sim} \upsilon$. Hence (p_*, E, q_*) is a right cofibration from B to A.

Finally, to see that $(\alpha e)\beta \xrightarrow{\sim} \alpha(e\beta)$, observe that



is a pullback in B^2; conclude that

$$\begin{array}{ccc} (\alpha e)\beta & \longrightarrow & e\beta \\ \downarrow & & \downarrow \\ a & \xrightarrow[\alpha]{} & ep \end{array}$$

is a pullback in E and note that this pullback defines $\alpha(e\beta)$. □

Remark 35. The proof of Proposition 34 also shows that in $\mathrm{TOP}^{\mathrm{op}}$ associativity follows from the other requirements for a fibration. That is: (p_*, E, q_*) is a TOP cofibration from B to A if and only if it is both a TOP left cofibration from B to A and a TOP right cofibration from B to A. □

In view of Proposition 29 for our generic $q: E \to B$, we might consider the left action of B on E to be the leading aspect and following Paré and Schumacher [5] write $\beta^*(e)$ for $\beta \circ e$ and $\Sigma_\beta(e)$ for $e\beta$. Then the B-indexed category corresponding to q has particularly well-behaved B-indexed sums. (There is some justification for this since $\mathrm{cod}: B^2 \to B$ provides the paradigmatic example of q and gives rise to the usual indexing of a base category B. We hasten to add, however, that geometric morphisms are usually taken to define quite a different indexing.) From this point of view it is natural to enquire whether the $@_*$ enjoyed by $p: E \to A$ endows the corresponding A-indexed category with any A-indexed products. In a meagre way the answer is yes.

To understand $@_*$ this way it suffices to understand $@_* u$. For e in E, write $\bar{e}$ for $e @_* u$. Then $(\bar{\ }): E \to E$ is right adjoint to $!(\ ): E \to E$ and it is easy to verify that $\bar{e} \simeq \overline{!e}$ (and $!e \simeq !\bar{e}$). So it suffices to study $\bar{e}$ for e such that $ep = 0$. In this case $\bar{e} \simeq \Pi_!(e)$, where $!$ is $0 \to \bar{e}p$, follows immediately from the adjunction.

6. Examples

In principle, Theorem 24 and Proposition 34 make the construction of TOP cofibrations a simple matter but we wish to emphasize two naturally occurring classes of examples.

The homomorphism $(\hat{\ }): \mathrm{CAT}^{\mathrm{coop}} \to \mathrm{CAT}$, defined by $\hat{E} = \mathrm{set}^{E^{\mathrm{op}}}$, has a left adjoint (viz. $(\check{\ })$ where $\check{F} = (\mathrm{set}^F)^{\mathrm{op}}$). Its restriction to $\mathrm{cat}^{\mathrm{coop}}$, where $\mathrm{cat} = \mathrm{cat(set)}$, factors through $\mathrm{TOP}^{\mathrm{op}} \to \mathrm{CAT}$ so $\mathrm{cat}^{\mathrm{co}} \to \mathrm{TOP}$ preserves the colimits needed to define cofibrations. CAT cofibrations were characterized by Street in terms of gamuts relative to $\mathrm{CAT} \to \mathrm{PROF}$. The reader is referred to [6].

Proposition 36. *If* $p: A \to E \leftarrow B: q$ *is a* cat *cofibration from* A *to* B, *then*

$p_*: \hat{A} \to \hat{E} \leftarrow \hat{B}: q_*$ *is a* TOP *cofibration from* B *to* A. *(Here* $\hat{p} \dashv p_*$ *and* $\hat{q} \dashv q_*$.) □

After Proposition 32, 33 and 34 the following topological result is immediate.

Proposition 37. *If* $F \rightarrowtail X \leftarrowtail U$ *in* top, *the category of small topological spaces, with* F *closed,* U *open and* $F \cap U = \emptyset$; *then* $\tilde{F} \rightarrowtail \tilde{X} \leftarrowtail \tilde{U}$ *is a* TOP *cofibration from* $\tilde{U}$ *to* $\tilde{F}$. □

7. Prononsense

If $\Phi: B \to A$ is a profunctor between categories, it is well known that the associated codiscrete cofibration is $A \to \underline{\Phi} \leftarrow B$, where $|\underline{\Phi}| = |A| + |B|$, $\underline{\Phi}(a', a) = A(a', a)$, $\underline{\Phi}(b', b) = B(b', b)$, $\underline{\Phi}(a, b) = \Phi(a, b)$, $\underline{\Phi}(b, a) = \emptyset$. This construction has a universal property relative to CAT→PROF. Indeed, arrows of CAT cospans,

$$(A \to \underline{\Phi} \leftarrow B) \to (f: A \to M \to B: g),$$

are in bijective correspondence with transformations, $\Phi \to g_* \otimes f^*$, in PROF. After Corollary 25 it should be clear that the glueing construction applied to a single left exact functor enjoys the same universal property relative to TOP→LEX^{co}.

Similar to the above construction for profunctors is that which associates to a protriple, $\Phi: A \to A$, a functor, $A \to A_\Phi$. Here $|A_\Phi| = |A|$, $A_\Phi(a', a) = \Phi(a', a)$, composition is accomplished via $\Phi \otimes \Phi \to \Phi$, and $A \to \Phi$ is used to define $A \to A_\Phi$. This construction too has an obvious universal property and it makes sense to enquire after it for any proarrow equipment, $(\ )_*$. Following the situation in CAT, it is reasonable to say that $f: A \to B$ in $\mathscr{K}$ is *bijective on objects relative to* $(\ )_*$ if B is equivalent to A_Φ for some triple Φ in $\mathscr{M}$. Necessarily Φ can be taken to be $f_* \otimes f^*$. Since a triple in LEX^{co} is a left exact cotriple, it is easy to see that the 'glueing' construction, which associates to a left exact cotriple the geometric morphism determined by the cofree functor, realizes the above universal property relative to TOP→LEX^{co}. In [9], $f: A \to B$ in $\mathscr{K}$ is defined to be *fully faithful relative to* $(\ )_*: \mathscr{K} \to \mathscr{M}$ if $\bar{f}: A \to f_* \otimes f^*$ is an isomorphism.

Theorem 38. *The image factorization system of* TOP *is the bijective on objects – fully faithful factorization system relative to* TOP→LEX^{co}. □

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